Generalized FFT Beamsteering

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Introduction: A receive antenna's beam, the direction of maximum gain or sensitivity, need not be rigidly oriented to the physical antenna. An array antenna, hundreds or thousands of small antenna elements laid out along a line or, more usefully, on a 2D lattice, offers an electronically controlled, agile beam or even multiple beams associated with multiple antenna outputs, a technology that can enable a military radar to precisely track many objects in flight simultaneously. The classic approach to efficiently realizing many simultaneous beams uses fast Fourier transforms (FFTs) for computational efficiency but severely restricts choices of system parameters. Generalized FFT beamsteering as described here loosens those restrictions to give system designers a larger tradeoff space and therefore opportunities for cost savings and performance gains, particularly for large digital radar arrays.

Classic Array Basics: The simplest array has antenna elements spaced along a line at half-wavelength intervals. Referring to Eq. (1) in Fig. 1, suppose the pth element output is processed by its own receiver to create complex signal output s_p (a function of time, though that is not shown here). Array output or beam sum X of Eq. (1) can be made to strongly favor signals from array boresight directions normal to the array, by designing weights w_p so that such signals add constructively in the sum while other signals largely cancel. Here p formally ranges from $-\infty$ to ∞ in the sum, but weight w_p is nonzero only for the finite number of p for which actual array elements exist.

Beam steering can be included in the beam-sum computation using the fact that a nonboresight signal arrives at the elements in sequence, with net propagation delays stepped element by element. For narrowband signals, which are more or less sinusoidal, stepped phase shifts result, so we aim the beam sum by modifying it to cancel those phase shifts for some direction of interest as in Eq. (2) of Fig. 1, where phase-step parameter θ selects the favored direction. To simultaneously create beams for N different θ values spread around the circle and thereby spread beams across all directions, specialize Eq. (2) to $\theta = 2\,\theta\pi k/N$ as in Eq. (3) and compute a beam sum X_k for each of $k = 0 \ldots N-1$.

This computation can be restructured to practical advantage. First, write each p as $Nn' + n$ with n' an arbitrary integer but with the existence and uniqueness of the decomposition guaranteed by drawing n only from the N modulo-N integers, classically taken to be $0 \ldots N-1$. Then define intermediate quantities x_n as in Eq. (4) and rewrite beam sum X_k of Eq. (3) as shown in Eq. (5).

This classic formulation expresses the N beam sums X_k as the size-N discrete Fourier transform (DFT) of the N intermediate variables x_n, which are precomputed using weighting and folding. This formulation makes the extraordinary computational efficiency of FFT algorithms available for computing the DFT. (For

$X = \sum_p w_p s_p$	boresight beam	(1)
$X = \sum_p w_p s_p e^{-j\theta p}$	steered beam	(2)
$X_k = \sum_p w_p s_p e^{-j2\pi kp/N}$	many steered beams $k = 0 \ldots N-1$	(3)
$x_n \triangleq \sum_{n'} w_{Nn'+n} s_{Nn'+n}$	weighted and folded element-receiver outputs	(4)
$X_k = \sum_{n \in \left\{ \substack{\text{all mod } N \\ \text{values}} \right\}} x_n e^{-j2\pi k N^{-1} n}$	FFT or generalized FFT to compute beams of (3)	(5)
$R = \begin{bmatrix} r_{\text{rows}} & 0 \\ 0 & r_{\text{cols}} \end{bmatrix}$	determines fine beam grid within coarse-grid cell	(6)
$N = \begin{bmatrix} -1 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 8 & 0 \\ 0 & 8 \end{bmatrix}$	example factoring to fix generalized-FFT structure	(7)

FIGURE 1
The key mathematics. Here Eq. (7) is for the Fig. 2 example. The other equations are general.

FFT basics see Ref. 1 or any other digital signal processing text.)

This FFT beamsteering approach is well known, as is its separate application in each of two dimensions to create beam rows and columns from a 2D array of element rows and columns.

NRL Generalizes to a Matrix FFT Size: Our work generalizes the latter 2D approach to add design flexibility. The subscripts on s_p, w_p, x_n, and X_k become length-two integer vectors, with k a row vector and the others columns. The original number of directions N becomes a nonsingular 2×2 integer matrix with the number of directions now just the magnitude $|N|$ of its determinant.

When N was a scalar, the set of "all mod-N values" summed over in Eq. (5) of Fig. 1 was any fixed set of N integers that ensured a unique decomposition $p = Nn' + n$ for every integer p, with $n = 0 \ldots N - 1$ the customary choice. A similar unique decomposition is required now as well, but now $p = Nn' + n$ involves matrix N and vectors n, n', and p, and n is drawn from any fixed set of $|N|$ modulo-N column vectors. Similarly, k now ranges over a set of modulo-N row vectors that gives every length-two integer row vector a unique decomposition $k'N + k$. In Eq. (5), exponent $kN^{-1}n$ is still a scalar but is now the product of a row vector, a square matrix, and a column vector.

Remarkably, if the array elements are laid out on a regular lattice, whether square, triangular, or neither, Eq. (5) of Fig. 1 with all these generalizations still creates a family of $|N|$ distinct beams and the principles underlying the classic FFT here lead to analogous algorithms. The efficiency of classic FFTs arises through factoring common factors out of sums and systematically sharing intermediate results. That is just as true here, and the results are just as computationally powerful.

The modulo-N ideas extend cleanly to the vector-matrix case because their scalar and vector versions both derive fundamentally from the same ideas in elementary group theory. Likewise, classic FFT algorithms and the generalized FFT algorithms of this work are both special cases of fast Fourier transforms on groups, a well-developed area in the mathematics community.

More Classics—The Grating-Lobe Offset Lattice: Beam-pointing parameter θ in Eq. (2) of Fig. 1 is a length-two vector with coordinates plotted horizontally and vertically in Fig. 2. In that θ space the globe shown has latitude and longitude lines corresponding respectively to a direction's elevation and azimuth relative to array boresight. A visible beam, one with θ inside the globe, is a signal direction of arrival favored in the corresponding beam sum. An invisible beam, one with θ outside the globe, is irrelevant to the array's physical operation.

There are always invisible beams in the mathematics, however, because beam sum X in Eq. (2) of Fig. 1 is periodic in θ. If X has a beam characterized by a particular θ, then it also has one characterized by θ plus

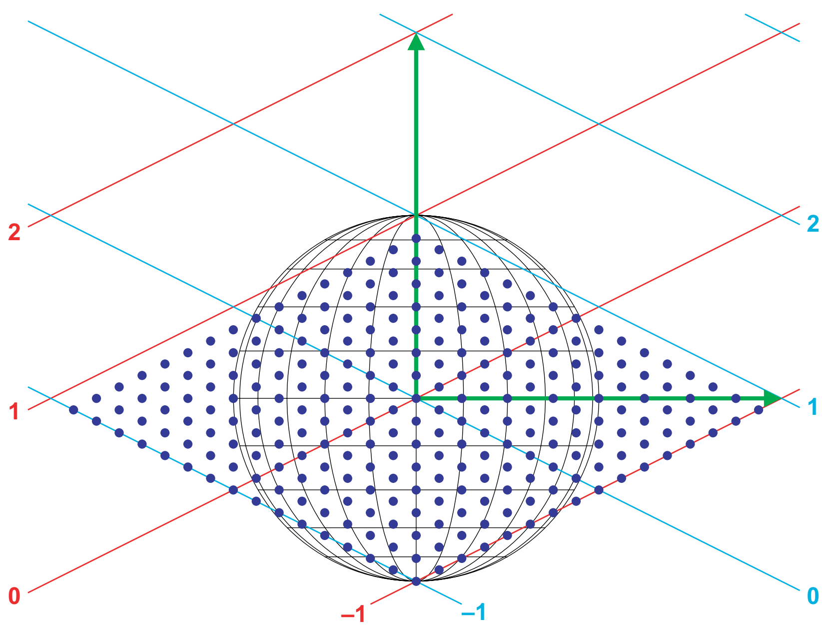

FIGURE 2
The θ plane. Dots in the direction-of-arrival coordinate globe represent beam locations computed by the generalized FFT. The beam layout and the structure of that FFT is determined by the choice of a coarse grid, which must include the basis vectors (green) of the grating-lobe lattice.

any member of a grating-lobe offset lattice comprising integer-weighted combinations of the large basis vectors shown as arrows in Fig. 2. Element spacing and placement geometry are ordinarily chosen—this is where the classic half-wavelength spacing arises—so that one period of the grating-lobe offset lattice completely contains the visible globe. Then no periodic replica of a desired beam can ever become visible and thus become a grating lobe.

The Increased Design Choice of the NRL Approach: In the design process a coarse grid is laid out in the θ space with grid lines at arbitrary orientations and angles—in Fig. 2 they make large diamonds—but such that the grating-lobe basis vectors fall on grid points with small integer coordinates, here $[-1\ \ 1]$ and $[2\ \ 2]$.

A matrix L is constructed with those two vectors as its rows. This will result in some (nonunique) set of $|L|$ grid cells exactly tiling the plane. Within each of those $|L|$ grid cells are r_{rows} rows by r_{cols} columns of θ values, here shown by dots, corresponding to beam directions realized in this scheme. The Fig. 2 example has $r_{rows} = r_{cols} = 8$. By varying these two parameters and how the coarse grid is chosen, many beam-position layouts are possible.

Matrix N is then constructed as $N = LR$, where R is a diagonal matrix of the r_{rows} and r_{cols} as in Eq. (6) of Fig. 1. Matrix N for the Fig. 2 example is shown in Eq. (7).

The $N = LR$ factorization yields a generalized FFT algorithm built from $|L|$ blocks, each built in turn from r_{rows} ordinary FFTs of size r_{cols} and r_{cols} ordinary FFTs of size r_{rows}. Complex-exponential "twiddle factors" (the traditional name) that were factored out of sums in the FFT derivation are applied to the block outputs, and the final array output to realize a beam then requires only a one-term-per-block linear combination of those twiddled block outputs. The computations leading to twiddled block outputs are shared across all beams. Only the final linear combinations of $|L|$ terms are specific to beams. (In the Fig. 2 example $|L| = 4$.) Values of θ outside the globe are irrelevant, as are those inside the globe but impractically close to its edge, so not all of the linear combinations need even be computed.

For a detailed treatment of this generalized FFT beamsteering technique, see Ref. 2.

[Sponsored by NRL]

References

[1] A.V. Oppenheim and R.W. Shafer, *Digital Signal Processing* (Prentice Hall, Englewood Cliffs, New Jersey, 1975).
[2] J.O. Coleman, "A Generalized {FFT} for Many Simultaneous Receive Beams," Naval Research Laboratory, NRL Memo Report 9029, June 29, 2007.