The provenance of modal inference

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Abstract

People reason about possibilities routinely, and reasoners can infer "modal" conclusions, i.e., conclusions that concern what is possible or necessary, from premises that make no mention of modality. For instance, given that *Cullen was born in New York or Kentucky*, it is intuitive to infer that *it's possible that Cullen was born in New York,* and a recent set of studies on modal reasoning bear out these intuitions (Hinterecker, Knauff, & Johnson-Laird, 2016). What explains the tendency to make modal inferences? Conventional logic does not apply to modal reasoning, and so logicians invented many alternative systems of modal logic to capture valid modal inferences. But, none of those systems can explain the inference above. We posit a novel theory based on the idea that reasoners build mental models, i.e., iconic simulations of possibilities, when they reason about sentential connectives such as *and*, *if*, and *or* (Johnson-Laird, 2006). The theory posits that reasoners represent a set of conjunctive possibilities to capture the meanings of compound assertions. It is implemented in a new computational process model of sentential reasoning that can draw modal conclusions from non-modal premises. We describe the theory and computational model, and show how its performance matches reasoners' inferences in two studies by Hinterecker et al. (2016). We conclude by discussing the model-based theory in light of alternative accounts of reasoning.

Keywords: mental models, modal reasoning, possibilities, reasoning, probability logic

Introduction

The word "possibility" is fraught with ambiguity, because philosophers distinguish between different sorts of possibility. An "alethic" possibility is any description that is not self-contradictory. "Deontic" possibilities are those that are permissible (e.g., instances of drinking alcohol when over the legal age restriction), and impossibilities are those that are prohibited (e.g., drinking while under the age restriction). Deontic possibilities can be violated, whereas alethic possibilities cannot (cf., Bucciarelli & Johnson-Laird, 2005; Bucciarelli, Khemlani, & Johnson-Laird, 2008). The present paper focuses on a different notion of possibility: "epistemic" possibilities concern possibilities that are consistent with a reasoner's personal knowledge. Reasoning based on possibilities is referred to as "modal" reasoning, because when you assert that something is "possible", you qualify its occurrence. Conventional systems of logic cannot take into account the logical properties of modals to draw conclusions, because they concern unqualified propositions that are either true or else false. Consider these two assertions:

1a. Sarah is Egyptian.
 b. Possibly, Sarah is Egyptian.

Assertion (1a) is unqualified: it asserts a fact about Sarah. If it is true then Sarah is indeed an Egyptian, and if it is false, she is not an Egyptian. Assertion (1b) is subtler. In addition to facts and their negations, it introduces possibilities. Logicians have historically analyzed modal assertions as referring to a set of "possible words" (see Kneale & Kneale, 1962; Portner, 2009). To say that something is *possibly* the case is to say that it is true in at least one possible world, and to say that something is *necessarily* the case is to say that it is true in all possible worlds.

Many different systems of modal logic exist (Kaufmann, Condoravi, & Harizanov, 2006). Each adopts a different set of axioms that affect which inferences can be proved. Different axiom systems affect which modal inferences are *valid* and which are not (see, e.g., Kripke, 1963). An inference is valid if it yields a conclusion that is true in every case in which the premises are true (Jeffrey, 1981, p. 1). In principle, an infinite number of modal logics exists, but logicians tend to focus on the axioms themselves, which run in parallel with semantic assumptions about the accessibility of one possibile world from another (Kripke, 1963). For instance, the axiom:

$$\Box A \rightarrow A$$

where '□' is a symbol that stands for the logical notion of necessity, *A* is any proposition whatsoever, and '→' denotes material implication. The axiom asserts that the necessity of *A* material implies *A*. The axiom does not hold in the modal logic "system K" (for "Kripke"), but it does hold another logic, "system T", and it corresponds to the assumption that accessibility is reflexive, i.e., if a proposition is necessary in a world then it holds in that world.

Here is a set of inferences that are invalid in all systems of modal logic:

2a. A or B or both.
 b. Therefore, possibly A.
 c. Therefore, possibly B.
 d. Therefore, possibly A and B.

These inferences seem intuitively reasonable, but the conclusions (2b-d) are invalid in any modal logic. Suppose that *A* is impossible but *B* is true. In logic, the premise is true, but (2b) is false. So, the inference is invalid. Similar suppositions show that all the inferences are invalid, and so, no modal logic permits them. Why, then, are the inferences compelling – almost "obvious" – for humans?

Despite some investigations into reasoning about possibilities (e.g., Bell & Johnson-Laird, 1998; Byrnes & Beilin, 1991; Goldvarg & Johnson-Laird, 2000; Inhelder & Piaget, 1958; Piéraut-Le Bonniec, 1980; Osherson, 1976; Sophian & Somerville, 1988), no comprehensive theory of human reasoning exists that explains how humans integrate reasoning about facts with reasoning about possibilities. The fundamental mystery is: where do the possibilities come from? Anecdotally, some researchers find that when participants are allowed to write or type out their own responses to a set of reasoning problems, they spontaneously qualify their inferences, e.g., by noting that a conclusion "is possible" or "could be true" or "might follow." These responses are, of course, different ways of expressing modal conclusions. Other research finds that reasoners are capable of carrying out various modal reasoning tasks systematically, e.g., given a set of premises, they are able to determine whether a conclusion is necessary or possible (e.g., Bell & Johnson-Laird, 1998; Khemlani, Lotstein, Trafton, & Johnson-Laird, 2015; Newstead & Griggs, 1983). But only recently have researchers examined reasoners' tendency to endorse modal conclusions from non-modal premises. Hinterecker and colleagues (2016) gave participants a battery of problems in which participants had to endorse or reject different conclusions from modal premises. Contrast this inference:

3. A or B or both.
   Therefore, possibly A and B.

with this one:

4. A or B, but not both.
   Therefore, possibly A and B.

Reasoners responded sensibly: they accepted (3) most of the time (82% of trials) but they accepted (4) on only a small minority of trials (10%; see Hinterecker et al., 2016, Experiment 1). But, both (3) and (4) are invalid in logic.

Probabilistic logic

Can Hinterecker and colleagues' findings be explained by an alternative theory? The two inferences above may be treated more sensibly in probabilistic logic, hereafter, "p-logic", which is a formal system devised by Adams (1975; 1988). P-logic reinterprets validity on probabilistic terms: a conclusion is probabilistically valid (p-valid) only if in any consistent assignment of probabilities its conclusion is at least as probable as its premises. Hence, in (3), the conclusion, *possibly(A and B)*, does not rule out any cases, i.e., it can be true independent of whether *A* or *B* are true. The premise, *A or B or both*, in contrast, rules out the situation in which both *A* and *B* are false. And so, the conclusion has a probability greater than that of the premise in (3), and it is p-valid. In a similar vein, (4) is p-invalid because the probability of the conclusion, *possibly(A and B)*, is 0 given the premise. And so, no matter what probability is assigned to the premise, the inference is p-invalid. P-logic is central to recent probabilistic accounts of human reasoning known colloquially as the "new paradigm" (see, e.g., Evans, 2012; Oaksford & Chater, 2007; Over, 2009; Johnson-Laird, Khemlani, & Goodwin, 2015).

But, p-logic does not always make sensible predictions. For instance, it predicts that the following inference is p-valid:

5. A or B, but not both.
   Therefore, A or B, or both.

The probability of the conclusion in (5) is greater than or equal to that of the premise, and so p-logic predicts that reasoners should make it. (The inference is always valid in logic.) Yet, participants rejected it on 97% of trials in the aforementioned study by Hinterecker and colleagues. Perhaps a deeper problem with probabilistic accounts is that they do not explain the provenance of modal conclusions, e.g., "possibly A", from non-modal premises. Hence, an alternative account of reasoning is needed to explain modal inferences.

A model-based theory of modal inference

The mental model theory of reasoning – hereafter, the "model theory" – posits that reasoners draw conclusions by building and scanning iconic representations of possibilities, i.e., mental models (Johnson-Laird, 2006; Johnson-Laird & Byrne, 1991). The theory assumes that interpreting compound assertions such as those linked by the connectives *and*, *or*, and *if*, yields a set of discrete possibilities. Models mimic the structure of what they represent, i.e., they are iconic (Peirce, 1931-1958, Vol. 4). But, they can also contain abstract tokens, such as symbols denoting negations (Khemlani, Orenes, & Johnson-Laird, 2012). They can represent temporal sequences of events as multiple models unfold in time the way events do (Bucciarelli, Mackiewicz, Khemlani, & Johnson-Laird, under review; Khemlani, Mackiewicz, Bucciarelli, & Johnson-Laird, 2013).

The theory posits two primary systems for reasoning (see, e.g., Johnson-Laird & Steedman, 1978): a fast system builds *mental models* and scans them without the use of working memory. A slower system revises models and fleshes them out to yield a set of *fully-explicit models*. It also searches for alternative models consistent with the premises. It can correct the errors and biases that the fast system yields, but it is subject to the limitations of working memory. The difference between mental models and fully-explicit models is clear when reasoning about disjunctions, e.g., *He has the* *soup or the salad or both.* Mental models abide by a "principle of truth", i.e., they represent what is true in a compound clause, and not what is false. They can flesh out the initial mental models to yield a set of *fully-explicit models*, i.e., possibilities that specify both what is true and what is false. The mental models of the disjunction above can be depicted in the following schematic diagram:

```
soup
          salad
soup      salad
```

Each row in the diagram denotes a different possibility. Hence, the first row denotes the possibility in which he has the soup. In contrast, a fully-explicit model represents both what is true in each possibility, as well as what is false:

```
  soup    ¬ salad
¬ soup      salad
  soup      salad
```

Three primary findings support the model theory. First, inferences from one model are easier than inferences from multiple models (e.g., Johnson-Laird, Byrne, & Schaeken, 1992). Second, because reasoners tend to build mental models instead of fully-explicit models, they are prone to systematic errors (see Khemlani & Johnson-Laird, 2017, for a review). Third, reasoners rely on counterexamples to correct erroneous inferences (e.g., Johnson-Laird & Hasson, 2003).

But, the theory has at least two serious shortcomings. First, it does not integrate facts and possibilities. As a result, it cannot explain the "obvious" inferences in example (2) above. Indeed, no theory of reasoning adequately integrates facts and modal reasoning, but the problem is particularly acute for the model theory, as the theory is based on the representation of possibilities, and so modal reasoning is within its purview. And second, its various computer implementations do not make quantitative predictions (Johnson-Laird & Yang, 2008). To rectify these shortcomings, we describe a novel assumption about the representation of mental models below, and then we present a new computational model capable of delivering quantitative predictions by varying how models are built and revised.

The principle of conjunctive possibilities

We amend the model theory to explain where possibilities come from in inferences that make no mention of them with the following principle:

> **The principle of conjunctive possibilities:** By default, compound assertions between clauses refer to conjunctions of possibilities. A clause can be evaluated as *possible* if it is affirmed in at least one possibility of the conjunctive set. It can be evaluated as *necessary* if it can be affirmed in all possibilities. And it is deemed *factual* if it is affirmed in a set of only one possibility.

The principle posits that a disjunction, *He has the soup or the salad or both*, refers to a set of possibilities, i.e.:

```
possible( soup & ¬salad ) &
possible(¬soup &  salad ) &
possible( soup &  salad )
```

The addition of the principle solves two mysteries of modal reasoning: first, it explains why reasoners are apt to make modal inferences from non-modal assertions. If compound assertions refer to possibilities, then reasoning about possibilities is the default instead of an extension to more basic reasoning patterns (cf. Inhelder & Piaget, 1958). Second, because the principle is that possibilities are related through conjunction, it allows reasoners to conclude that any of the separate possibilities can be concluded as *possible*. An immediate consequence of the assumption is that modal inferences are the default, and reasoning about facts is a special case of reasoning about possibilities.

The principle is presaged by recent ideas due to Zimmerman (2000), who proposed that disjunctions refer to lists of alternatives in a "possible worlds" semantics, and Geurts (2005) who extended the idea to disjunctions that concern facts. The principle we propose, however, applies to all sorts of sentential connectives, including disjunctions, conjunctions, conditionals, and even causal relations, e.g., *causes*, *enables*, and *prevents* (Johnson-Laird & Khemlani, in press; Khemlani, Barbey, & Johnson-Laird, 2014).

The principle maintains the separation between mental models and fully-explicit models. Hence, it makes all of the same predictions as previous versions of the model theory. It also predicts that reasoners should deem (5) invalid, which we repeat here:

5. A or B, but not both.
   Therefore, A or B, or both.

Both a truth-functional analysis in logic and the notion of p-validity in p-logic treats (5) as valid. But, if reasoners represent the exclusive disjunction as a conjunction of possibilities, i.e.:

```
possible( soup & ¬salad ) &
possible(¬soup &  salad )
```

then the conclusion does not follow from the representation, because nothing yields the possibility in which both cases hold.

Nevertheless, the previous predictions are qualitative, not quantitative. A veridical simulation of human reasoning needs to provide a quantitative simulation of the extant data. To do so, we developed a novel computational implementation of the model theory, and we tested it against two experiments by Hinterecker et al. (2016). We now describe the computational model and its simulation of data.

A computational implementation of the model theory

We developed a computational theory of sentential reasoning that integrates reasoning about facts and reas-

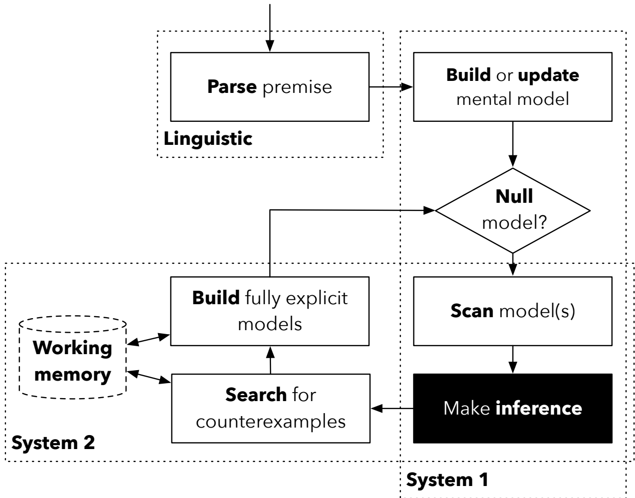

Figure 1. A schematic diagram of the computational model of reasoning. The system operates by parsing premises in natural language, constructing mental models and scanning them to formulate initial conclusions (system 1), and then searching for counterexamples and building fully-explicit models to interrogate initial inferences (system 2).

oning about possibilities. It implements the principles of the mental model theory of reasoning (see, e.g., Khemlani & Johnson-Laird, 2013) and the principle of conjunctive possibilities introduced here. Figure 1 provides a schematic of the program. The computational model is structured around three general systems:

a) A linguistic system uses a grammar and lexicon to parse verbal assertions.
b) An intuitive system (System 1) uses the parse to construct an initial mental model, i.e., a conjunction of possibilities. It also scans the model to formulate initial inferences.
c) A deliberative system (System 2) can flesh out the mental model and search for alternative models. This system can manipulate and update the representations created in System 1, and it can modify conclusions, but it too can fail when a problem calls for more working memory than it has (Khemlani & Johnson-Laird, 2017).

In the computational model, system 1 does not have access to working memory, and so it can construct only one mental model at a time. It can flesh out the mental model to make it explicit. The probability of doing so is governed by a parameter, φ. System 2, however, has access to working memory. As a result, the operations of system 1 are faster and more prone to err than system 2. System 2 can operate on multiple models at a time, search for counterexamples, and construct a set of fully explicit models. The probability of calling system 2 is governed by a separate parameter, σ. In principle, the size of working memory could also be governed by a parameter in order to model individual differences in reasoning.

The system is capable of carrying out a number of inferential tasks, but for brevity, we consider just two: assessing that a given conclusion is possible, and assessing that it is necessary. In order to assess an inference, the computer model checks that each possibility in the conjunction corresponding to the conclusion is supported by the premises. If they all are, then the conclusion follows of necessity, and if at least one is, then the conclusion is a possibility. The concept of necessity may vary from participant to participant and from problem to problem: some problems may encourage reasoners to check that the models of the premises hold in all models of the conclusion (a strong notion of validity), and some problems may encourage reasoners to check only that all the possibilities to which the conclusions refer hold in the premises (a weaker notion of validity). Consider how you might respond to problems such as this one:

6. Suppose that: A or B, or both.
   Does it follow that: A or B, but not both?

The conjunctive possibilities to which the first premise refers are:

```
possible( A & ¬B ) &
possible(¬A &  B ) &
possible( A &  B )
```

And the possibilities to which the second premise refers are:

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possible( A & ¬B ) &
possible(¬A &  B )
```

Reasoners with a strong notion of validity should judge that (6) is invalid, because the models of the premise do not all hold in the models of the conclusion. Reasoners with a weaker notion of validity should assess that (6) is valid, since the models of the conclusion hold in all the models of the premises. To simulate this non-determinism of human reasoning, we built a third and final parameter into the system, γ, that denotes the probability of the system adopting a weak version of validity.

We applied the computational model to simulate recent data on modal reasoning. The simulations show a close fit between the predictions of the system and humans' inferential behavior after conservative parameter searching.

Simulations of Hinterecker et al. (2016)

We sought to use the computational implementation of the model theory to simulate participants' performance in Experiments 1 and 3 in Hinterecker et al. (2016), as those two studies are directly pertinent to how reasoners draw modal conclusions from non-modal premises. In Experiment 1, Hinterecker and colleagues gave participants a battery of diagnostic problems that involve disjunctions. Two of those problems tested the tendency to draw modal conclusions from disjunctions that make no mention of possibilities, and two tested the ability to infer an inclusive disjunction from an exclusive one, and an exclusive

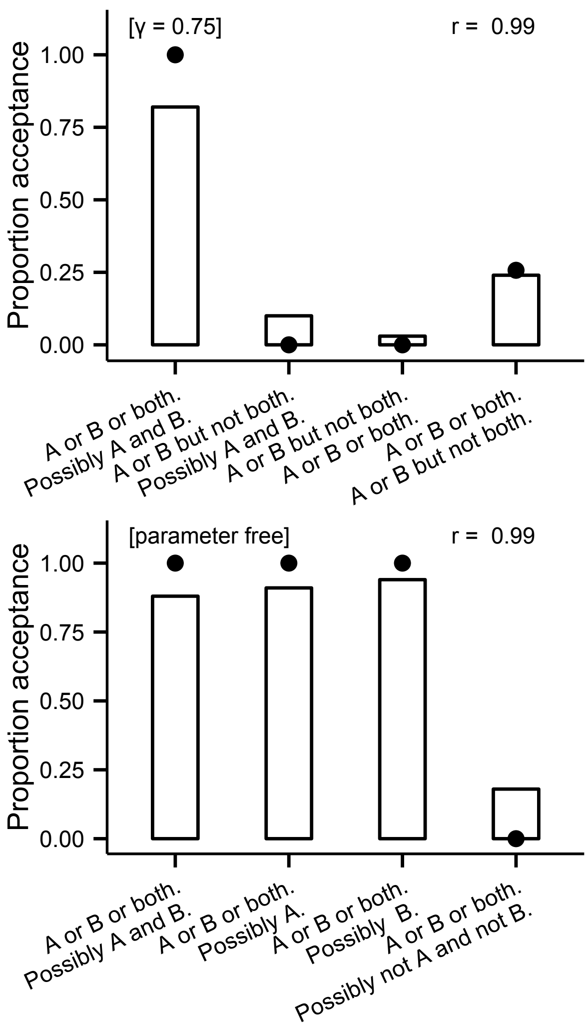

Figure 2. Observed (histograms) data and predicted (circles) proportions of accepting the conclusion for different inferences. *Top panel:* inferences from Hinterecker et al. (2016, Experiment 1). *Bottom panel:* inferences from Hinterecker et al. (2016, Experiment 2). For each of the problems, the assertion on top denotes the premise and the assertion on the bottom denotes the conclusion.

disjunction from an inclusive one. Figure 2 summarizes the proportion of participants to accept the varying conclusions given the single premise.

The only parameter that could have affected the system's simulations on the problems in Experiment 1 was the γ parameter, which dictates how probable it is for participants to make use of a weak notion of validity. An exhaustive exploration of the parameter space yielded an optimal γ value of .75, i.e., the system optimally modeled the data when it stochastically applied weak validity to 75% of simulated problems. We generated synthetic data by running 1000 simulations of the four inferences in Experiment 1. Figure 2 (top panel) shows the proportion of correct responses in the observations (histograms) and predictions (circles) in the study as a function of the inference. The computer model matched the participants' performance in the experiment well (r = .99, RMSE = .10). The predictions of the computer model were in the 99th percentile relative to hypothetical datasets (Khemlani & Trafton, 2013).

Hinterecker's et al. (2016) Experiment 3 was a more stringent test of reasoners' ability to infer modal conclusions from an inclusive disjunction, *A or B or both*. For each problem in the experiment, participants assessed the disjunction and then accepted or rejected the one of the following four conclusions: *possibly A and B, possibly A, possibly B, possibly not-A and not-B.* None of these inferences is valid in any known logic, but as Figure 2 (bottom panel) shows, reasoners endorsed three of the four conclusions. We disabled all of the parameters to see how the computer model matched the participants' performance; it did so extremely well (r = .99, RMSE = .12), and additional parameter manipulations would have resulted in only nominal changes to the fit.

In sum, the computational model implementing the model theory and the principle of conjunctive possibilities yielded a close fit to the data from Hinterecker et al. (2016).

General discussion

Reasoners have no difficulty drawing modal conclusions from compound assertions that make no mention of modality. The ability to do so often seems "obvious"; only experts are likely to realize that this inference is invalid in logic:

7. A or B or both.
   Therefore, possibly A.

No known logical system designed to deal with modalities, i.e., a modal logic concerning what is possible or necessary, permits the inference above. Reasoners naive to logic may also be surprised to find that both orthodox logic and probabilistic logic render the following inference valid:

8. A or B, but not both.
   Therefore, A or B, or both.

On our account, reasoners are justified in feeling that the invalidity of (7) and the validity of (8) are counterintuitive and incorrect. The model theory of reasoning, which is based on possibilities (Johnson-Laird, 2006), treats compound assertions, such as conjunctions, conditionals, and the disjunctions in (7) and (8), as conjunctions of possibilities. Hence, reasoning about possibilities is fundamental. Reasoners represent possibilities directly, and so modal reasoning is a natural consequence of the way people represent assertions.

The principle of conjunctive possibilities characterizes the inferences in (6) and (7) more intuitively: it predicts that (6) should be deemed valid and (7) should be deemed invalid. And a computational implementation of the principle makes identical predictions, which are validated by recent experiments on modal reasoning by Hinterecker et al. (2016).

At present, no alternative theory of reasoning, whether based on mental logic (e.g., Rips, 1994) or on the probability calculus (e.g., Oaksford & Chater, 2007), can explain these phenomena of modal reasoning. Moreover, no computational model of reasoning, whether in psychology, artificial intelligence, or logic, characterizes the inferences in the same manner as the system we outlined above. The reason, as we argue, is that everyday reasoning is based on possibilities, not probabilities or truth-functions.

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