Changes from 2013 NRL Plasma Formulary to the
2016 NRL Plasma Formulary

▷ p. 32

from

$$= 6.8 \times 10^{-8} \frac{\mu'^{1/2}}{\mu} \left(1 + \frac{\mu'}{\mu}\right)^{-1/2} T^{-3/2}$$

to

$$= 6.8 \times 10^{-8} \frac{\mu'^{1/2}}{\mu} \left(1 + \frac{\mu'}{\mu}\right) T^{-3/2}$$

▷ p. 37

'frictional force' and 'electrical conductivities' formulae changed to

$$\mathbf{R_u} = \frac{ne}{\sigma_0}(0.51\mathbf{j}_\parallel + \mathbf{j}_\perp); \qquad \sigma_0 = ne^2\tau_e/m_e$$

This reflects the definition of $\mathbf{R_u}$ in the Braginskii formulation. Earlier versions of the NRL plasma formulary referred to the variables $\sigma_\parallel$ and $\sigma_\perp$ as 'conductivities' or 'electrical conductivities' in the definition of the frictional force $\mathbf{R_u}$. This has caused some confusion (brought to my attention by several plasma physicists) and the above formula should dispel any confusion.

▷ p. 34

The expression for electron-ion collisons (b) in the cold electron limit should be

$$\lambda_{ei} = \lambda_{ie} = 16 - \ln\left(n_i^{1/2} T_i^{-3/2} Z^2 \mu\right)$$

for $T_e < T_i(m_e/m_i)$.

▷ p. 35

The factor 35 in the expression for $\lambda_{ii'}$ has been changed to 43.

Coulomb logarithm for counter-streaming ions in the presence of warm electrons

Scott C. Hsu
Physics Division, Los Alamos National Laboratory, Los Alamos, NM 87545

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We re-derive the Coulomb logarithm for counter-streaming ions in the presence of warm electrons and obtain a different result than the one given in the NRL Plasma Formulary (2013 edition).

I. RE-DERIVATION OF THE COUNTER-STREAMING ION-ION COULOMB LOGARITHM

The Coulomb logarithm for counter-streaming ions with relative velocity $v_D = \beta_D c$ in the presence of warm electrons ($kT_i/m_i, kT_{i'}/m_{i'} < v_D^2 < kT_e/m_e$) is given in the NRL Plasma Formulary (2013 edition, top of p. 35) as

$$\lambda_{ii'} = \lambda_{i'i} = 35 - \ln\left[\frac{ZZ'(\mu+\mu')}{\mu\mu'\beta_D^2}\left(\frac{n_e}{T_e}\right)^{1/2}\right], \qquad (1)$$

where T_e is in eV and cgs units are used throughout, consistent with the convention in the NRL Plasma Formulary. Unprimed and primed variables correspond to test and field particles, respectively.

In some cases that are clearly weakly coupled, $\lambda_{ii'}$ falls below unity as calculated using Eq. (1). For example, for counter-streaming Al-Al collisions with $\mu = \mu' = 27$, $Z = Z' = 2.0$, $v_D = 20$ km/s, $T_e = 2.2$ eV, and $n_e = 6.5 \times 10^{14}$ cm^{-3}, $\lambda_{ii} = 0.325$. This prompted us to check Eq. (1) for accuracy.

The definition of the Coulomb logarithm is

$$\lambda = \ln\Lambda = \ln\left(\frac{r_{\max}}{r_{\min}}\right), \qquad (2)$$

where in this case

$$r_{\max} = \lambda_{De} = \left(\frac{kT_e}{4\pi n_e e^2}\right)^{1/2} = 7.43\times 10^2\left(\frac{T_e}{n_e}\right)^{1/2} \text{ [cm]} \qquad (3)$$

and

$$r_{\min} = b = \frac{ZZ'e^2}{m_{ii'}v_D^2} \text{ [cm]}, \qquad (4)$$

where b the distance of closest approach between two counter-streaming ions with reduced mass $m_{ii'} = m_i m_{i'}/(m_i + m_{i'})$ and relative speed v_D. Here, we assume that b is greater than the de Broglie wavelength $\hbar/m_{ii'}v_D$.

We re-write b by pulling out numerical constants:

$$b = \frac{e^2}{m_p c^2}\frac{(\mu+\mu')ZZ'}{\mu\mu'(v_D/c)^2} = \frac{(4.8032\times10^{-10})^2}{(1.6726\times10^{-24})(2.9979\times10^{10})^2}\frac{(\mu+\mu')ZZ'}{\mu\mu'\beta_D^2} = 1.5347\times10^{-16}\frac{(\mu+\mu')ZZ'}{\mu\mu'\beta_D^2}. \qquad (5)$$

Plugging Eqs. (3) and (5) into Eq. (2), we obtain

$$\lambda_{ii'} = \ln\Lambda = \ln\left[\frac{743(T_e/n_e)^{1/2}}{1.5347\times10^{-16}\frac{(\mu+\mu')ZZ'}{\mu\mu'\beta_D^2}}\right] = 43 - \ln\left[\frac{ZZ'(\mu+\mu')}{\mu\mu'\beta_D^2}\left(\frac{n_e}{T_e}\right)^{1/2}\right]. \qquad (6)$$

The discrepancy between the numerical factors 43 [in Eq. (6)] and 35 [in Eq. (1)] is exactly accounted for if the constants $k = 1.6022\times10^{-12}$ erg/eV and $e^2 = (4.8032\times10^{-10})^2$ statcoul^2 in Eq. (3) are ignored (thanks to G. Swadling of Imperial College for pointing this out), which would change the numerical factor of 743 to 0.282 in Eqs. (3) and (6). This seems like a plausible mistake to make in the original derivation of Eq. (1).

Using the Al-Al parameters given earlier, we get $b = 1.02\times10^{-8}$ cm, $\lambda_{De} = 4.32\times10^{-5}$ cm, and $\lambda_{ii} = \ln\Lambda = 8.1$, a more reasonable result for our weakly coupled plasma.

In summary, we propose that Eq. (6) is the more correct expression for the Coulomb logarithm for counter-streaming ions in the presence of warm electrons.

Coulomb Logarithm
NRL Formulary p. 34
J.D. Huba (04 April 2016)

$$\lambda = \ln \Lambda = \ln \frac{r_{\max}}{r_{\min}} \tag{1}$$

$r_{\min}$ is defined as

$$r_{\min} = \frac{e_\alpha e_\beta}{m_{\alpha\beta} \bar{u}^2} \tag{2}$$

where

$$m_{\alpha\beta} = \frac{m_\alpha m_\beta}{m_\alpha + m_\beta} \quad \text{and} \quad \bar{\mathbf{u}} = \mathbf{v}_\alpha - \mathbf{v}_\beta$$

for $\alpha = i$ (ion) and $\beta = e$ (electron) and from Justin Ball find that

$$m_{ie} = m_e \quad \text{and} \quad \bar{u}^2 = \frac{3kT_i}{m_i} \tag{3}$$

where k is the Boltzmann constant and assumed that $T_e < T_i(m_e/m_i)$. Thus,

$$r_{\min} = \frac{Ze^2}{3kT_i} \frac{m_i}{m_e} \tag{4}$$

$r_{\max}$ is defined as

$$r_{\max} = \left(4\pi \sum \frac{n_\gamma e_\gamma^2}{kT_\gamma} \right)^{-1/2} \tag{5}$$

which we write as

$$r_{\max} = \left(\frac{kT_i}{4\pi n_i Z^2 e^2} \right)^{1/2} \tag{6}$$

using (4) and (6) obtain

$$\ln \Lambda = \ln \left(\frac{kT_i}{4\pi n_i Z^2 e^2} \right)^{1/2} \frac{3kT_i}{Ze^2} \frac{m_e}{m_i} \tag{7}$$

$$\ln \Lambda = \ln \left(\frac{3k^{3/2}}{2\pi^{1/2} e^3} \frac{m_e}{m_i} \right) \frac{T_i^{3/2}}{\mu Z^2 n_i^{1/2}} \tag{8}$$

$$\ln\Lambda = \ln\left(\frac{3k^{3/2}}{2\pi^{1/2}e^3}\frac{m_e}{m_i}\right) - \ln\left(\frac{\mu Z^2 n_i^{1/2}}{T_i^{3/2}}\right) \tag{9}$$

calculate constant (first term above). first,

$$k = (1.38\times10^{-16})(1.16\times10^4) = 2.02\times10^{-18}$$

where 1.16×10^4 is a conversion to eV

$$e^3 = (4.80\times10^{-10})^3 = 1.11\times10^{-28}$$

$$\frac{m_e}{m_i} = 5.46\times10^{-4}$$

so we get

$$\frac{3k^{3/2}}{2\pi^{1/2}e^3}\frac{m_e}{m_i} = 8.41\times10^6$$

and so

$$\ln\left(\frac{3k^{3/2}}{2\pi^{1/2}e^3}\frac{m_e}{m_i}\right) = \ln(8.41\times10^6) = 15.94 \simeq 16$$

so on p. 34 of NRL plasma formulary should have

$$\lambda_{ei} = \lambda_{ie} = 16 - \ln\left(n_i^{1/2}T_i^{-3/2}Z^2\mu\right) \tag{10}$$

for $T_e < T_i(m_e/m_i)$.

Justin Ball
02 April 2016

For Electron-ion collisions

$$\ln\Lambda \equiv \ln\left(\frac{r_{max}}{r_{min}}\right)$$

$$r_{max} \equiv \left(\sum_s' \frac{q_s^2 n_s}{\epsilon_0 T_s}\right)^{-1/2} \text{ where sum is over species with } \bar{u}^2 \lesssim \frac{T_s}{m_s}$$

$$\bar{u}^2 = \frac{1}{n_i n_e}\int d^3v \int d^3v' \, |\vec{v}-\vec{v}'|^2 f_{Mi}(\vec{v}) f_{Me}(\vec{v}')$$

$$\bar{u}^2 = \frac{3T_i}{m_i} + \frac{3T_e}{m_e} \qquad \text{where } f_{Ms}(\vec{v}) \equiv n_s\left(\frac{m_s}{2\pi T_s}\right)^{3/2}\exp\left(\frac{-m_s v^2}{2T_s}\right)$$

$$\bar{u}^2 = \frac{3T_i}{m_i} \text{ because } \boxed{T_e < T_i\frac{m_e}{m_i}} \Rightarrow \frac{T_e}{m_e} < \frac{T_i}{m_i}$$

$$r_{max} = \sqrt{\frac{\epsilon_0 T_i}{Z^2 e^2 n_i}} \quad \text{because} \quad \bar{u}^2 = \frac{3T_i}{m_i} > \frac{T_e}{m_e}$$

$$r_{min} = \text{Max}\left[\frac{Ze^2}{4\pi\epsilon_0 m_r \bar{u}^2}, \frac{\hbar}{2m_r\bar{u}}\right] \text{ where } m_r \equiv \frac{m_i m_e}{m_i + m_e} \to m_e$$

$$r_{min} = \frac{Ze^2}{4\pi\epsilon_0 m_e}\frac{m_i}{3T_i}\text{Max}\left[1, \frac{\hbar}{2m_e}\sqrt{\frac{m_i}{3T_i}}\frac{4\pi\epsilon_0 m_e}{Ze^2}\frac{3T_i}{m_i}\right]$$

$$r_{min} = \frac{Ze^2}{4\pi\epsilon_0 m_e}\frac{m_i}{3T_i}\text{Max}\left[1, \frac{2\sqrt{3}\pi\epsilon_0\hbar}{e^2\sqrt{m_e}}\sqrt{\frac{T_i m_e}{Z^2 m_i}}\right]$$

$$r_{min} = \frac{Ze^2}{4\pi\epsilon_0 m_e}\frac{m_i}{3T_i}\text{Max}\left[1, \frac{2\sqrt{30}\pi\epsilon_0\hbar}{e^2\sqrt{m_e}}\sqrt{1\,eV}\sqrt{\frac{1}{10Z^2}\frac{T_i m_e}{(1\,eV)m_i}}\right]$$

$$r_{min} = \frac{Ze^2}{4\pi\epsilon_0 m_e}\frac{m_i}{3T_i}\text{Max}\left[1, 0.5\sqrt{\frac{1}{10Z^2}\frac{T_i m_e}{(1\,eV)m_i}}\right]$$

$$r_{min} = \frac{Ze^2}{4\pi\epsilon_0 m_e}\frac{m_i}{3T_i} \quad \text{for} \quad \boxed{T_i\frac{m_e}{m_i} < 40 Z^2\, eV}$$

$$\ln\Lambda = \ln\left(\frac{1}{Ze}\sqrt{\frac{\epsilon_0 T_i}{n_i}}\frac{12\pi\epsilon_0 m_e T_i}{Ze^2 m_i}\right)$$

$$\ln\Lambda = \ln\left(\frac{T_i^{3/2} m_p}{Z^2\sqrt{n_i}\, m_i}\frac{12\pi\epsilon_0^{3/2} m_e}{e^3 m_p}\right)$$

$$\ln\Lambda = \ln\left(\frac{12\pi\epsilon_0^{3/2} m_e}{e^3 m_p}\frac{(eV)^{3/2}}{\sqrt{cm^{-3}}}\right) - \ln\left(\sqrt{\frac{n_i}{cm^{-3}}}\left(\frac{T_i}{eV}\right)^{-3/2} Z^2\mu\right)$$

$$\boxed{\ln\Lambda = 16 - \ln\left(\sqrt{\frac{n_i}{cm^{-3}}}\left(\frac{T_i}{eV}\right)^{-3/2} Z^2\mu\right)} \qquad \text{where } \mu \equiv \frac{m_i}{m_p}$$

from Reviews of Plasma Physics (Consultants Bureau, New York, 1965)
ed. M.A. Leontovich, p. 105
'Particle Interactions in a Fully Ionized Plasma'
B.A. Trubnikov

eq 18.5 (p. 189)

$$\tau_s^{\alpha/\beta} = \frac{\tau_l^{\alpha/\beta}}{\left(1 + \frac{m_\alpha}{m_\beta}\right)\mu} \tag{1}$$

where

$$\mu = \mu(x_\beta), \quad x_\beta = \frac{\epsilon_\beta}{T_\beta} = \frac{m_\beta}{m_\alpha}\frac{\epsilon_\alpha}{T_\beta}, \quad \tau_l^{\alpha/\beta} = \left(\frac{m_\alpha}{2}\right)^{1/2} \frac{1}{\pi e_\alpha^2 e_\beta^2} \frac{\epsilon_\alpha^{3/2}}{\lambda n_\beta}$$

limiting cases for $\mu(x)$ (p. 177, eq 15.10)

$$\mu(x) = \begin{cases} \frac{4x^{3/2}}{3(\pi)^{1/2}} & x << 1 \\ 1 & x >> 1 \end{cases}$$

1. fast case ($x_\beta >> 1$):

$$\mu \simeq 1, \quad \tau_l^{\alpha/\beta} \sim m_\alpha^{1/2}\epsilon_\alpha^{3/2}$$

substitute into Eq. (1)

$$\tau_s^{\alpha/\beta} \propto \frac{m_\alpha^{1/2}}{\left(1 + \frac{m_\alpha}{m_\beta}\right)} = \frac{1}{m_\alpha^{1/2}}\left(\frac{1}{m_\alpha} + \frac{1}{m_\beta}\right)^{-1} \tag{2}$$

$$\nu_s^{\alpha/\beta} = \frac{1}{\tau_s^{\alpha/\beta}} \;\propto\; m_\alpha^{1/2}\left(\frac{1}{m_\alpha} + \frac{1}{m_\beta}\right) \tag{3}$$

set $m_\alpha = \mu$ and $m_\beta = \mu'$ and obtain

$$\nu_s^{\mu/\mu'} \;\propto\; m_\mu^{1/2}\left(\frac{1}{m_\mu} + \frac{1}{m'_\mu}\right) \tag{4}$$

2. slow case ($x_\beta << 1$):

$$\mu(x) \simeq x^{3/2}, \quad \tau_l^{\alpha/\beta} \sim m_\alpha^{1/2}$$

substitute into Eq. (1)

$$\tau_s^{\alpha/\beta} \propto \left(\frac{m_\beta}{m_\alpha}\right)^{-3/2} \frac{m_\alpha^{1/2}}{\left(1+\frac{m_\alpha}{m_\beta}\right)} = \left(\frac{m_\alpha}{m_\beta}\right)^{1/2} \frac{m_\alpha}{m_\beta} \frac{m_\alpha^{1/2}}{\left(1+\frac{m_\alpha}{m_\beta}\right)} = \frac{m_\alpha}{m_\beta^{1/2}} \frac{1}{\left(1+\frac{m_\beta}{m_\alpha}\right)}$$

thus,

$$\nu_s^{\alpha/\beta} = \frac{1}{\tau_s^{\alpha/\beta}} \propto \frac{m_\beta^{1/2}}{m_\alpha} \left(1+\frac{m_\beta}{m_\alpha}\right)$$

and

$$\nu_s^{\mu/\mu'} \propto \frac{\mu'^{1/2}}{\mu} \left(1+\frac{\mu'}{\mu}\right)$$