Changes from 2013 NRL Plasma Formulary to the 2016 NRL Plasma Formulary

\[ \frac{\mu^{1/2}}{\mu} \left( 1 + \frac{\mu'}{\mu} \right)^{-1/2} T^{-3/2} \]

\[ \frac{\mu^{1/2}}{\mu} \left( 1 + \frac{\mu'}{\mu} \right) T^{-3/2} \]

'frictional force' and 'electrical conductivities' formulae changed to

\[ \mathbf{R_u} = \frac{n_e}{\sigma_0} (0.51 \mathbf{j}_\parallel + \mathbf{j}_\perp) \]

\[ \sigma_0 = n e^2 \tau_e / m_e \]

This reflects the definition of \( \mathbf{R_u} \) in the Braginskii formulation. Earlier versions of the NRL plasma formulary referred to the variables \( \sigma_\parallel \) and \( \sigma_\perp \) as 'conductivities' or 'electrical conductivities' in the definition of the frictional force \( \mathbf{R_u} \). This has caused some confusion (brought to my attention by several plasma physicists) and the above formula should dispel any confusion.

\[ \lambda_{ei} = \lambda_{ie} = 16 - \ln \left( n_i^{1/2} T_i^{-3/2} Z^2 \mu \right) \]

for \( T_e < T_i (m_e/m_i) \).

The factor 35 in the expression for \( \lambda_{ii} \) has been changed to 43.
Coulomb logarithm for counter-streaming ions in the presence of warm electrons

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We re-derive the Coulomb logarithm for counter-streaming ions in the presence of warm electrons and obtain a different result than the one given in the NRL Plasma Formulary (2013 edition).

I. RE-DERIVATION OF THE COUNTER-STREAMING ION-ION COULOMB LOGARITHM

The Coulomb logarithm for counter-streaming ions with relative velocity \( v_D = \beta_D e \) in the presence of warm electrons \( (kT_i/m_i, kT_i'/m_i' < v_D^2 < kT_e/m_e) \) is given in the NRL Plasma Formulary (2013 edition, top of p. 35) as

\[
\lambda_{ii'} = \lambda_{i'i} = 35 - \ln \left[ \frac{ZZ' (\mu + \mu')}{\mu\mu'\beta_D^2} \left( \frac{n_e}{T_e} \right)^{1/2} \right],
\]

(1)

where \( T_e \) is in eV and cgs units are used throughout, consistent with the convention in the NRL Plasma Formulary. Unprimed and primed variables correspond to test and field particles, respectively.

In some cases that are clearly weakly coupled, \( \lambda_{ii'} \) falls below unity as calculated using Eq. (1). For example, for counter-streaming Al-Al collisions with \( \mu = \mu' = 27 \), \( Z = Z' = 20 \), \( v_D = 20 \) km/s, \( T_e = 2.2 \) eV, and \( n_e = 6.5 \times 10^{14} \) cm\(^{-3} \), \( \lambda_{ii} = 0.325 \). This prompted us to check Eq. (1) for accuracy.

The definition of the Coulomb logarithm is

\[
\lambda = \ln \Lambda = \ln \left( \frac{r_{\text{max}}}{r_{\text{min}}} \right),
\]

(2)

where in this case

\[
r_{\text{max}} = \lambda_D e = \left( \frac{kT_e}{4\pi n_e e^2} \right)^{1/2} = 7.43 \times 10^2 \left( \frac{T_e}{n_e} \right)^{1/2} [\text{cm}]
\]

(3)

and

\[
r_{\text{min}} = b = \frac{ZZ' e^2}{m_{ii'} v_D^2} [\text{cm}],
\]

(4)

where \( b \) the distance of closest approach between two counter-streaming ions with reduced mass \( m_{ii'} = m_i m_i' / (m_i + m_i') \) and relative speed \( v_D \). Here, we assume that \( b \) is greater than the de Broglie wavelength \( \hbar/m_{ii'} v_D \).

We re-write \( b \) by pulling out numerical constants:

\[
b = \frac{e^2}{m_e c^2} \frac{(\mu + \mu') ZZ'}{\mu\mu'(v_D/c)^2} = \frac{(4.8032 \times 10^{-10})^2}{(1.6726 \times 10^{-24})(2.9979 \times 10^{10})^2} \frac{(\mu + \mu') ZZ'}{\mu\mu'\beta_D^2} = 1.5347 \times 10^{-16} \frac{(\mu + \mu') ZZ'}{\mu\mu'\beta_D^2}.
\]

(5)

Plugging Eqs. (3) and (5) into Eq. (2), we obtain

\[
\lambda_{ii'} = \ln \Lambda = \ln \left[ \frac{743(T_e/n_e)^{1/2}}{1.5347 \times 10^{-16} (\mu + \mu') ZZ' (\mu\mu'\beta_D^2)} \right] = 43 - \ln \left[ \frac{ZZ' (\mu + \mu')}{\mu\mu'\beta_D^2} \left( \frac{n_e}{T_e} \right)^{1/2} \right].
\]

(6)

The discrepancy between the numerical factors 43 [in Eq. (6)] and 35 [in Eq. (1)] is exactly accounted for if the constants \( k = 1.6022 \times 10^{-12} \) erg/eV and \( e^2 = (4.8032 \times 10^{-10})^2 \) statcoul\(^2 \) in Eq. (3) are ignored (thanks to G. Swadling of Imperial College for pointing this out), which would change the numerical factor of 743 to 0.282 in Eqs. (3) and (6). This seems like a plausible mistake to make in the original derivation of Eq. (1).

Using the Al-Al parameters given earlier, we get \( b = 1.02 \times 10^{-8} \) cm, \( \lambda_D e = 4.32 \times 10^{-5} \) cm, and \( \lambda_{ii} = \ln \Lambda = 8.1 \), a more reasonable result for our weakly coupled plasma.

In summary, we propose that Eq. (6) is the more correct expression for the Coulomb logarithm for counter-streaming ions in the presence of warm electrons.
\( \lambda = \ln \Lambda = \ln \frac{r_{\text{max}}}{r_{\text{min}}} \) \hspace{1cm} (1)

\( r_{\text{min}} \) is defined as

\[ r_{\text{min}} = \frac{e_\alpha e_\beta}{m_{\alpha\beta} \bar{u}^2} \] \hspace{1cm} (2)

where

\[ m_{\alpha\beta} = \frac{m_\alpha m_\beta}{m_\alpha + m_\beta} \] \hspace{0.5cm} \text{and} \hspace{0.5cm} \bar{u} = v_\alpha - v_\beta

for \( \alpha = i \) (ion) and \( \beta = e \) (electron) and from Justin Ball find that

\[ m_{ie} = m_e \] \hspace{0.5cm} and \hspace{0.5cm} \( \bar{u}^2 = \frac{3kT_i}{m_i} \] \hspace{1cm} (3)

where \( k \) is the Boltzmann constant and assumed that \( T_e < T_i(m_e/m_i) \). Thus,

\[ r_{\text{min}} = \frac{Ze^2}{3kT_i m_e} \] \hspace{1cm} (4)

\( r_{\text{max}} \) is defined as

\[ r_{\text{max}} = \left( 4\pi \sum n_\gamma e_\gamma^2 \frac{k}{kT_\gamma} \right)^{-1/2} \] \hspace{1cm} (5)

which we write as

\[ r_{\text{max}} = \left( \frac{kT_i}{4\pi n_i Z^2 e^2} \right)^{1/2} \] \hspace{1cm} (6)

using (4) and (6) obtain

\[ \ln \Lambda = \ln \left( \frac{kT_i^{3/2}}{4\pi n_i Z^2 e^2} \right)^{1/2} \frac{3kT_i m_e}{Ze^2 m_i} \] \hspace{1cm} (7)

\[ \ln \Lambda = \ln \left( \frac{3k^{3/2}}{2\pi^{1/2} e^3 m_i} \right) \frac{T_i^{3/2}}{\mu Z^2 n_i 1/2} \] \hspace{1cm} (8)
\[ \ln \Lambda = \ln \left( \frac{3k^{3/2} m_e}{2\pi^{1/2} e^3 m_i} \right) - \ln \left( \frac{\mu Z^2 n_i^{1/2}}{T_i^{3/2}} \right) \tag{9} \]

calculate constant (first term above). first,

\[ k = (1.38 \times 10^{-16})(1.16 \times 10^4) = 2.02 \times 10^{-18} \]

where \(1.16 \times 10^4\) is a conversion to eV

\[ e^3 = (4.80 \times 10^{-10})^3 = 1.11 \times 10^{-28} \]

\[ \frac{m_e}{m_i} = 5.46 \times 10^{-4} \]

so we get

\[ \frac{3k^{3/2} m_e}{2\pi^{1/2} e^3 m_i} = 8.41 \times 10^6 \]

and so

\[ \ln \left( \frac{3k^{3/2} m_e}{2\pi^{1/2} e^3 m_i} \right) = \ln(8.41 \times 10^6) = 15.94 \simeq 16 \]

so on p. 34 of NRL plasma formulary should have

\[ \lambda_{ei} = \lambda_{ie} = 16 - \ln \left( n_i^{1/2} T_i^{-3/2} Z^2 \mu \right) \tag{10} \]

for \( T_e < T_i (m_e/m_i) \).
For electron-ion collisions

\[ \ln \Lambda = \ln \left( \frac{\Gamma_{\text{max}}}{\Gamma_{\text{min}}} \right) \]

\[ \Gamma_{\text{max}} = \left( \sum_{s} \frac{Z_{s}^{2} n_{s} e_{s}}{e_{o} T_{s}} \right)^{1/2} \]

where sum is over species with \( u^{2} \leq \frac{T_{e}}{m_{i}} \)

\[ u^{2} = \frac{1}{n_{i} m_{i}} \int d^{3}v \int d^{3} v' \left| \hat{v} - \hat{v}' \right|^{2} f_{\text{ext}}(\hat{v}) f_{\text{me}}(\hat{v}') \]

\[ \left( \frac{u}{u_{i}} \right)^{2} = \frac{3T_{i}}{m_{i}} + \frac{3T_{e}}{m_{e}} \]

where \( f_{\text{me}}(\hat{v}) = n_{s} \left( \frac{m_{s}}{2 \pi m_{i}} \right)^{3/2} \exp \left( \frac{-m_{s} v^{2}}{2 m_{i}} \right) \)

\[ u^{2} = \frac{3T_{i}}{m_{i}} \text{ because } \frac{T_{e}}{T_{i}} \ll \frac{m_{e}}{m_{i}} \Rightarrow \frac{T_{e}}{m_{e}} < \frac{T_{i}}{m_{i}} \]

\[ \Gamma_{\text{max}} = \sqrt{\frac{E_{0} T_{s}}{Z_{e}^{2} n_{e}}} \text{ because } \frac{1}{u^{2}} = \frac{3T_{i}}{m_{i}} > \frac{T_{e}}{m_{e}} \]

\[ \Gamma_{\text{min}} = \max \left[ \frac{Z_{e}^{2}}{4 \pi e_{0} m_{e} m_{i} u^{2}}, \frac{2 e_{0}}{Z_{e}^{2} m_{i}} \right] \text{ where } m_{r} = \frac{e_{0} m_{e}}{2 e_{0} m_{e} + m_{i}} = m_{e} \]

\[ \gamma_{\text{min}} = \frac{Z_{e}^{2}}{4 \pi e_{0} m_{e}} \frac{m_{o}}{3 T_{i}} \max \left[ 1, \frac{2 \sqrt{3} \pi T_{e} e_{0}^{2}}{Z_{e}^{2} m_{i}} \right] \]

\[ \gamma_{\text{min}} = \frac{Z_{e}^{2}}{4 \pi e_{0} m_{e}} \frac{m_{o}}{3 T_{i}} \max \left[ 1, \frac{2 \sqrt{3} \pi T_{e} e_{0}^{2}}{Z_{e}^{2} m_{i}} \right] \]

\[ \gamma_{\text{min}} = \frac{Z_{e}^{2}}{4 \pi e_{0} m_{e}} \frac{m_{o}}{3 T_{i}} \max \left[ 1, \frac{2 \sqrt{3} \pi T_{e} e_{0}^{2}}{Z_{e}^{2} m_{i}} \right] \]

\[ \gamma_{\text{min}} = \frac{Z_{e}^{2}}{4 \pi e_{0} m_{e}} \frac{m_{o}}{3 T_{i}} \text{ for } \frac{T_{i}}{m_{i}}^{2/3} < 40 Z_{e}^{2} \text{ eV} \]

\[ \ln \Lambda = \ln \left( \frac{\frac{1}{Z_{e}^{2}} \frac{E_{0} T_{s}}{n_{i} m_{i}}}{\frac{1}{Z_{e}^{2}} \frac{E_{0} m_{e}}{m_{i}} e_{0}^{2} m_{i}} \right) \]

\[ \ln \Lambda = \ln \left( \frac{\frac{1}{Z_{e}^{2}} \frac{E_{0} m_{e}}{m_{i}} e_{0}^{2} m_{i}}{\frac{1}{Z_{e}^{2}} m_{e} e_{0}^{2} m_{i}} \right) \]

\[ \ln \Lambda = \ln \left( \frac{\frac{1}{Z_{e}^{2}} \frac{E_{0}^{3/2} m_{e}}{m_{i}} e_{0}^{3/2} m_{i}}{\frac{1}{Z_{e}^{2}} m_{e} e_{0}^{3/2} m_{i}} \right) \]
eq 18.5 (p. 189)

\[ \tau_s^{\alpha/\beta} = \frac{\tau_l^{\alpha/\beta}}{(1 + \frac{m_\alpha}{m_\beta}) \mu} \]  

where

\[ \mu = \mu(x_\beta), \quad x_\beta = \frac{\epsilon_\beta}{T_\beta} = m_\beta \frac{\epsilon_\alpha}{m_\alpha T_\beta}, \quad \tau_l^{\alpha/\beta} = \left( \frac{m_\alpha}{2} \right)^{1/2} \frac{1}{\pi e_\alpha^2 e_\beta^2} \frac{\epsilon_\alpha^{3/2}}{\Lambda \epsilon_\beta} \]

limiting cases for \( \mu(x) \) (p. 177, eq 15.10)

\[ \mu(x) = \begin{cases} 
4 \frac{x^{3/2}}{3(\pi)^{1/2}} & x << 1 \\
1 & x >> 1 
\end{cases} \]

1. fast case \( (x_\beta >> 1) \):

\[ \mu \simeq 1, \quad \tau_l^{\alpha/\beta} \sim m_\alpha^{1/2} \epsilon_\alpha^{3/2} \]

substitute into Eq. (1)

\[ \tau_s^{\alpha/\beta} \propto \frac{m_\alpha^{1/2}}{(1 + \frac{m_\alpha}{m_\beta})} = \frac{1}{m_\alpha^{1/2}} \left( \frac{1}{m_\alpha} + \frac{1}{m_\beta} \right)^{-1} \]  

\[ \nu_s^{\alpha/\beta} = \frac{1}{\tau_s^{\alpha/\beta}} \propto m_\alpha^{1/2} \left( \frac{1}{m_\alpha} + \frac{1}{m_\beta} \right) \]  

set \( m_\alpha = \mu \) and \( m_\beta = \mu' \) and obtain

\[ \nu_s^{\mu/\mu'} \propto m_\mu^{1/2} \left( \frac{1}{m_\mu} + \frac{1}{m_\mu'} \right) \]
2. slow case $x_\beta << 1$:

$$\mu(x) \simeq x^{3/2}, \quad \tau_{a/\beta}^a \sim m_\alpha^{1/2}$$

substitute into Eq. (1)

$$\tau_{a/\beta}^a \propto \left( \frac{m_\beta}{m_\alpha} \right)^{-3/2} \frac{m_\alpha^{1/2}}{1 + \frac{m_\alpha}{m_\beta}} = \left( \frac{m_\alpha}{m_\beta} \right)^{1/2} \frac{m_\alpha}{m_\beta} \left(1 + \frac{m_\alpha}{m_\beta}\right) = \frac{m_\alpha}{m_\beta} \left(1 + \frac{m_\beta}{m_\alpha}\right)$$

thus,

$$\nu_{a/\beta} = \frac{1}{\tau_{a/\beta}^a} \propto \frac{m_\alpha^{1/2}}{m_\alpha} \left(1 + \frac{m_\beta}{m_\alpha}\right)$$

and

$$\nu_{a/\mu} \propto \frac{\mu'^{1/2}}{\mu} \left(1 + \frac{\mu'}{\mu}\right)$$